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Journal of Sound and Vibration 295 (2006) 342–361
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

The exact solution for free vibration of uniform beams carrying multiple two-degree-of-freedom spring–mass systems

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Received 24 January 2005; received in revised form 22 December 2005; accepted 14 January 2006
Available online 11 April 2006

Abstract

The literature regarding the "exact" solutions of natural frequencies and mode shapes of a uniform beam carrying multiple two-degree-of-freedom (2-dof) spring–mass systems is rare, thus, this paper aims at studying this problem using the numerical assembly method (NAM). First of all, the equivalent springs for replacing the effect of a 2-dof spring–mass system are determined. Next, the coefficient matrix for a 2-dof spring–mass system attached to the uniform beam is derived based on the compatibility of deformations and equilibrium of forces (including moments). The coefficient matrices for the left end and right end of the beam are also derived based on the various boundary conditions of the beam. Combining the coefficient matrices for all the 2-dof spring–mass systems attached to the beam and the coefficient matrices for the boundary conditions of the beam, one obtains the overall coefficient matrix of the constrained beam (i.e., the beam carrying any number of 2-dof spring–mass systems). The product of the overall coefficient matrix and the vector for all the integration constants yields a set of simultaneous equations. Let the coefficient determinant of the last simultaneous equations equal to zero, one obtains the frequency equation. The roots of the last frequency equation denote the natural frequencies of the constrained beam. Substituting the roots of the frequency equation into the set of simultaneous equations one may determine the associated mode shapes of the constrained beam. In this paper, the "exact" solution refers to the one obtained from the "continuous" model instead of the "discrete" mode, besides, the accuracy of the analytical-and-numerical combined method (ANCM) given by the existing literature is dependent on the total number of vibration modes considered, but this is not true for the accuracy of the NAM adopted here. To confirm the reliability of the presented theory, all the numerical results obtained from NAM are compared with the corresponding ones obtained from the conventional finite element method (FEM) and good agreement is achieved.

1. Introduction

The free vibration problem about a "uniform" or "non-uniform" beam carrying elastically mounted lumped masses has been studied by a lot of researchers [1–12]. Jen and Magrab [13] and Chang and Chang [14], respectively, performed the free and force vibration analyses of the "uniform" beam carrying a two-degree-of-freedom (2-dof) spring–mass system using the Laplace transform with respect to the spatial variable. In 1999, Wu and Whittaker [15] studied the free vibration characteristics of the "uniform" beam carrying

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0022-460X/$ - see front matter
doi:10.1016/j.jsv.2006.01.011

multiple 2-dof spring–mass systems using the analytical-and-numerical-combined method (ANCM). In Refs. [9–11], the numerical assembly method (NAM) was used for the free vibration analysis of the "uniform" and "non-uniform" beams carrying multiple one-degree-of-freedom (1-dof) spring–mass systems. This is the first report that focuses on the free vibration analysis of a uniform beam carrying multiple 2-dof spring–mass systems by using the NAM. First of all, for either ANCM or NAM, each 2-dof spring–mass system must be replaced by two massless equivalent springs with effective spring constants $k_{\text{eff},i}$ and $k_{\text{eff},k}$ [15].

Since the accuracy of most approximate solutions is evaluated by using the corresponding "exact" ones, a lot of researchers devote themselves to the study of this aspect [1–6,16]. This is one of the reasons why the title problem is studied. In theory, the solutions of NAM adopted in this paper are the exact ones, because they are based on the continuous model rather than the discrete model used by the conventional finite element method (FEM) or the other approximate methods concerned. Furthermore, the formulation of ANCM adopted by Wu and Whittaker [15] is based on the natural frequencies and mode shapes of the "bare" beam (without carrying any attachment), its accuracy is dependent on the total number of vibration modes considered, but this is not true for the accuracy of NAM.

From the following sections of this paper, one finds that the eigenequation of the title problem takes the form $[B]\{\overline{C}\} = \{0\}$. Since the order of the overall coefficient matrix $[B]$ is $p = 8n + 4$, with n being the total number of attachments (i.e., 2-dof spring–mass systems), the order of $[B]$ is 12 for one attachment and 20 for two attachments. It is evident that the explicit expression for the eigenequation $[B]\{\overline{C}\} = \{0\}$ will become lengthy and complicated for the cases with $n > 2$, hence the literature relating to the free vibration analysis of a uniform beam carrying more than "two" concentrated attachments is rare.

Because the NAM presented in Refs. [9–11] has been found to be able to easily tackle the free vibration problem of the "uniform" and "non-uniform" beams carrying any number of 1-dof spring–mass systems, this paper tries to use the same approach to perform the free vibration analysis of a "uniform" beam carrying any number of 2-dof spring–mass systems.

To show the reliability of the introduced approach, the lowest five natural frequencies and some of the corresponding mode shapes of a uniform beam carrying five 2-dof spring–mass systems are calculated. Four boundary conditions are studied: clamped–free, clamped–simply supported, clamped–clamped, and simply supported–simply supported. It has been found that the agreement between the present results and the FEM results is good.

For convenience, the uniform beam with prescribed boundary conditions is called the "bare" beam if it carries no attachment and is called the "constrained" beam if it carries any attachments.

2. Replacing a 2-dof spring–mass system by two equivalent springs

For the 2-dof spring–mass system shown in Fig. 1(a), m_e and J_e, respectively, represent the lumped mass and mass moment of inertia of the system, k_1 and k_2 are the spring constants of the springs, u_w and θ_w are the

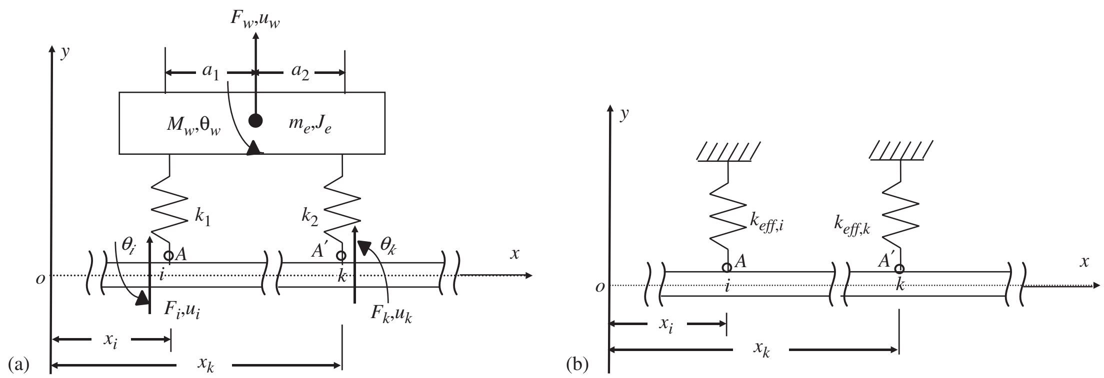

Fig. 1. (a) A uniform beam carrying a 2-dof spring–mass system and (b) replacing the 2-dof spring–mass system is by two equivalent springs $k_{\text{eff},i}$ and $k_{\text{eff},k}$.

translational and rotational displacements of the lumped mass m_e, a_1 and a_2 are distances between the center of gravity (c.g.) of the lumped mass and the two springs, u_i and θ_i are the transverse displacement and rotational angle of the uniform beam at the attaching node i, and u_k and θ_k are the those at the attaching node k. The upward force on the 2-dof spring–mass system at point A and point A' are given by

$$F_i(t) = k_1[u_i(t) + a_1\theta_w(t) - u_w(t)], \tag{1}$$

$$F_k(t) = k_2[u_k(t) - a_2\theta_w(t) - u_w(t)]. \tag{2}$$

The equations of motion of the 2-dof spring–mass system can be written as

$$m_e\ddot{u}_w + k_1(-u_i - a_1\theta_w + u_w) + k_2(-u_k + a_2\theta_w + u_w) = 0, \tag{3}$$

$$J_e\ddot{\theta}_w + k_1(a_1u_i + a_1^2\theta_w - a_1u_w) + k_2(-a_2u_k + a_2^2\theta_w + a_2u_w) = 0. \tag{4}$$

For free vibration of the constrained beam (i.e., the bare beam together with the 2-dof spring–mass system), one has

$$u_i(t) = \bar{u}_i e^{i\bar{\omega}t}, \quad u_k(t) = \bar{u}_k e^{i\bar{\omega}t}, \quad u_w(t) = \bar{u}_w e^{i\bar{\omega}t}, \quad \theta_w(t) = \bar{\theta}_w e^{i\bar{\omega}t}, \tag{5}$$

$$F_i(t) = \bar{F}_i e^{i\bar{\omega}t}, \quad F_k(t) = \bar{F}_k e^{i\bar{\omega}t}, \tag{6}$$

where $\bar{\omega}$ is the natural frequency of the constrained beam, while $\bar{u}_w$, $\bar{\theta}_w$, $\bar{u}_i$, $\bar{u}_k$, $\bar{F}_i$, and $\bar{F}_k$ represent the vibration amplitudes of $u_w(t)$, $\theta_w(t)$, $u_i(t)$, $u_k(t)$, $F_i(t)$ and $F_k(t)$, respectively.

The substitution of Eq. (5) into Eqs. (3) and (4), one obtains

$$\begin{bmatrix} m_e\bar{\omega}^2 - (k_1 + k_2) & a_1k_1 - a_2k_2 \\ a_1k_1 - a_2k_2 & J_e\bar{\omega}^2 - (k_1a_1^2 + k_2a_2^2) \end{bmatrix} \begin{Bmatrix} \bar{u}_w \\ \bar{\theta}_w \end{Bmatrix} = \begin{Bmatrix} -k_1\bar{u}_i - k_2\bar{u}_k \\ k_1a_1\bar{u}_i - k_2a_2\bar{u}_k \end{Bmatrix}. \tag{7}$$

For simplicity, Eq. (7) can also be rewritten as

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{Bmatrix} \bar{u}_w \\ \bar{\theta}_w \end{Bmatrix} = \begin{Bmatrix} -k_1\bar{u}_i - k_2\bar{u}_k \\ k_1a_1\bar{u}_i - k_2a_2\bar{u}_k \end{Bmatrix}, \tag{8}$$

where

$$A_{11} = m_e\bar{\omega}^2 - (k_1 + k_2), \tag{9}$$

$$A_{12} = A_{21} = a_1k_1 - a_2k_2, \tag{10}$$

$$A_{22} = J_e\bar{\omega}^2 - (k_1a_1^2 + k_2a_2^2). \tag{11}$$

Solving Eqs. (3) and (4) for $u_w(t)$ and $\theta_w(t)$, one obtains

$$u_w(t) = [(-A_{22}k_1 - A_{12}k_1a_1)u_i(t) - (A_{22}k_2 - A_{12}k_2a_2)u_k(t)]/(A_{11}A_{22} - A_{12}^2), \tag{12}$$

$$\theta_w(t) = [(A_{12}k_1 + A_{11}k_1a_1)u_i(t) + (A_{12}k_2 - A_{11}k_2a_2)u_k(t)]/(A_{11}A_{22} - A_{12}^2). \tag{13}$$

Substituting Eqs. (12) and (13) into the right-hand side of Eqs. (1) and (2) yields

$$\begin{Bmatrix} F_i(t) \\ F_k(t) \end{Bmatrix} = \begin{bmatrix} k_{\text{eff},11} & k_{\text{eff},12} \\ k_{\text{eff},21} & k_{\text{eff},22} \end{bmatrix} \begin{Bmatrix} u_i(t) \\ u_k(t) \end{Bmatrix}, \tag{14}$$

where

$$\begin{aligned} k_{\text{eff},11} &= (k_1 + 2A_{12}k_1^2a_1 + A_{11}k_1^2a_1^2 + A_{22}k_1^2)/(A_{11}A_{22} - A_{12}^2), \\ k_{\text{eff},12} &= k_{\text{eff},21} = (A_{12}k_1k_2a_1 - A_{11}k_1k_2a_1a_2 + A_{22}k_1k_2 - A_{12}k_1k_2a_2)/(A_{11}A_{22} - A_{12}^2), \\ k_{\text{eff},22} &= (k_2 - 2A_{12}k_2^2a_2 + A_{11}k_2^2a_2^2 + A_{22}k_2^2)/(A_{11}A_{22} - A_{12}^2). \end{aligned} \tag{15}$$

In order to apply the NAM to solve the title problem, the last four "effective" springs must be further replaced by two "equivalent" spring constants $k_{\mathrm{eff},i}$ and $k_{\mathrm{eff},k}$. The latter are found to be

$$k_{\mathrm{eff},i} = k_{\mathrm{eff},11} + k_{\mathrm{eff},12}u_k(t)/u_i(t), \quad k_{\mathrm{eff},k} = k_{\mathrm{eff},22} + k_{\mathrm{eff},21}u_i(t)/u_k(t). \tag{16,17}$$

The exact solution presented in the paper will agree with Chang et al. [14], if the effect of the coupling effective spring constants $k_{\mathrm{eff},12}$ and $k_{\mathrm{eff},21}$ defined by Eq. (15) is negligible.

3. Eigenfunctions for the constrained uniform beam

Fig. 2 shows a uniform cantilever beam carrying n 2-dof spring–mass systems. If the two attaching points of the vth 2-dof spring–mass system are, respectively, denoted by $i^{(v)}$ and $k^{(v)}$ (located at $x = x_i^{(v)}$ and $x = x_k^{(v)}$ $(v = 1, 2, \ldots, r)$), then, for the attaching point $i^{(v)}$, its "left side" belongs to the beam segment $i^{(v)}$ and its "right side" belongs to the beam segment $i^{(v)} + 1$ (or $k^{(v)}$). Similarly, for the attaching point $k^{(v)}$, its "left side" belongs to the beam segment $k^{(v)}$ (or $i^{(v)} + 1$) and its "right side" belongs to the beam segment $k^{(v)}$. For convenience, the left end and the right end of the beam are denoted by Ⓛ and Ⓡ, respectively. It is noted that nodes $i^{(v)}$ and $k^{(v)}$ are adjacent and node $k^{(v)}$ is next to node $i^{(v)}$, thus, the numbering of node $k^{(v)}$ is the same as that of node $i^{(v)} + 1$.

The equation of motion for a uniform beam is given by [16]

$$EI\frac{\partial^4 y(x,\, t)}{\partial x^4} + \rho A\frac{\partial^2 y(x,\, t)}{\partial t^2} = 0, \tag{18}$$

where $y(x,t)$ is the transverse deflection, E is the Young's modulus, A is the cross-sectional area, I is the moment of inertia of cross-sectional area A about the axis of bending, ρ is the mass density of the beam material and t is time.

For free vibration of the beam, one has

$$y(x,\, t) = \overline{Y}(x)\mathrm{e}^{\mathrm{i}\overline{\omega}t}, \tag{19}$$

where $\overline{\omega}$ is the natural frequency of the beam and $\overline{Y}(x)$ is the amplitude of $y(x,\, t)$.

The substitution of Eq. (19) into Eq. (18) yields

$$\overline{Y}''''(x) - \beta^4\overline{Y}(x) = 0, \tag{20}$$

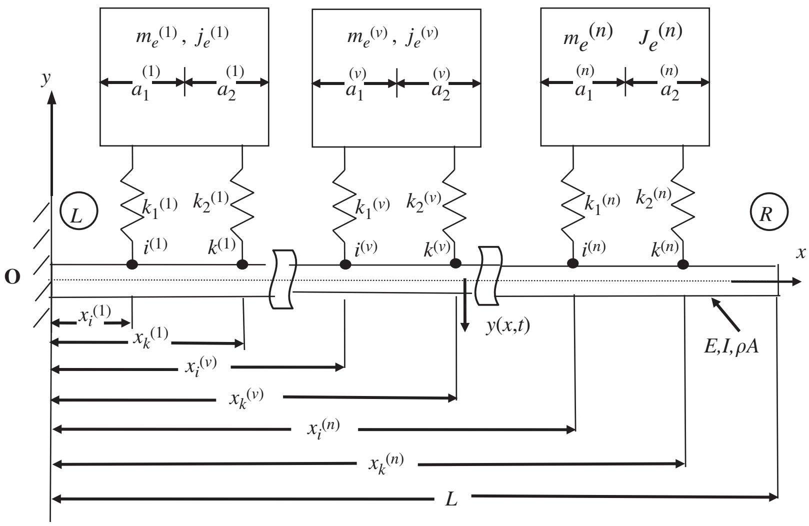

Fig. 2. A uniform cantilever beam carrying n 2-dof spring–mass systems.

where

$$\beta^4 = \frac{\rho A \overline{\omega}^2}{EI}. \tag{21}$$

The general solution of Eq. (20) takes the form

$$\overline{Y}(x) = C_1 \sin(\beta x) + C_2 \cos(\beta x) + C_3 \sinh(\beta x) + C_4 \cosh(\beta x), \tag{22}$$

where C_i $(i = 1, \ldots, 4)$ are the integration constants.

Eq. (22) represents the eigenfunction for the transverse deflection of the beam. Once the natural frequencies $\overline{\omega}_j$ $(j = 1, 2, \ldots)$ and the constants for each attaching point, C_i $(i = 1, \ldots, 4)$, are determined from the next sections, one may obtain the value of $\overline{Y}_j(x)$. The latter are the mode shapes of the beam corresponding to the natural frequency $\bar{\omega}_j$.

For the vth 2-dof spring–mass system with attaching points $i^{(v)}$ and $k^{(v)}$, the transverse deflections of "the beam segment $i^{(v)}$" and "the beam segment $k^{(v)}$" are given by (cf. Eq. (22))

$$\overline{Y}_i^{(v)}(x_i^{(v)}) = C_{i1}^{(v)} \sin(\beta x_i^{(v)}) + C_{i2}^{(v)} \cos(\beta x_i^{(v)}) + C_{i3}^{(v)} \sinh(\beta x_i^{(v)}) + C_{i4}^{(v)} \cosh(\beta x_i^{(v)}), \tag{23}$$

$$\overline{Y}_k^{(v)}(x_k^{(v)}) = C_{k1}^{(v)} \sin(\beta x_k^{(v)}) + C_{k2}^{(v)} \cos(\beta x_k^{(v)}) + C_{k3}^{(v)} \sinh(\beta x_k^{(v)}) + C_{k4}^{(v)} \cosh(\beta x_k^{(v)}). \tag{24}$$

4. Coefficient matrix $[B^{(v)}]$ for the vth 2-dof spring–mass system with attaching points $i^{(v)}$ and $k^{(v)}$

Compatibility of deflections and slopes together with equilibrium of bending moments and shearing forces at the attaching point $i^{(v)}$ requires that

$$\overline{Y}_i^{(v)L}(x_i^{(v)}) = \overline{Y}_i^{(v)R}(x_i^{(v)}), \tag{25a}$$

$$\overline{Y}_i'^{(v)L}(x_i^{(v)}) = \overline{Y}_i'^{(v)R}(x_i^{(v)}), \tag{25b}$$

$$\overline{Y}_i''^{(v)L}(x_i^{(v)}) = \overline{Y}_i''^{(v)R}(x_i^{(v)}), \tag{25c}$$

$$EI\overline{Y}_i'''^{(v)L}(x_i^{(v)}) - k_{\mathrm{eff},i}^{(v)} \overline{Y}_i^{(v)L}(x_i^{(v)}) = EL\overline{Y}_i'''^{(v)R}(x_i^{(v)}). \tag{25d}$$

Similarly, at the attaching point $k^{(v)}$ one has

$$\overline{Y}_k^{(v)L}(x_k^{(v)}) = \overline{Y}_k^{(v)R}(x_k^{(v)}), \tag{25e}$$

$$\overline{Y}_k'^{(v)L}(x_k^{(v)}) = \overline{Y}_k'^{(v)R}(x_k^{(v)}), \tag{25f}$$

$$\overline{Y''}_k^{(v)L}(x_k^{(v)}) = \overline{Y''}_k^{(v)R}(x_k^{(v)}), \tag{25g}$$

$$EI\overline{Y}_k'''^{(v)L}(x_k^{(v)}) - k_{\mathrm{eff},k}^{(v)} \overline{Y}_k^{(v)L}(x_k^{(v)}) = EI\overline{Y}_k'''^{(v)R}(x_k^{(v)}), \tag{25h}$$

where $k_{\mathrm{eff},i}^{(v)}$ and $k_{\mathrm{eff},k}^{(v)}$ are the equivalent springs constants for the vth 2-dof spring–mass system given by Eqs. (16) and (17).

Substituting Eqs. (23) and (24) into Eq. (25), one obtains

$$\begin{aligned} & C_{i1}^{(v)}\sin(\beta x_i^{(v)}) + C_{i2}^{(v)}\cos(\beta x_i^{(v)}) + C_{i3}^{(v)}\sinh(\beta x_i^{(v)}) + C_{i4}^{(v)}\cosh(\beta x_i^{(v)}) \\ & \quad - C_{i+1,1}^{(v)}\sin(\beta x_i^{(v)}) - C_{i+1,2}^{(v)}\cos(\beta x_i^{(v)}) - C_{i+1,3}^{(v)}\sinh(\beta x_i^{(v)}) - C_{i+1,4}^{(v)}\cosh(\beta x_i^{(v)}) = 0, \end{aligned} \tag{26}$$

$$\begin{aligned} & C_{i1}^{(v)}\beta\cos(\beta x_i^{(v)}) - C_{i2}^{(v)}\beta\sin(\beta x_i^{(v)}) + C_{i3}^{(v)}\beta\cosh(\beta x_i^{(v)}) + C_{i4}^{(v)}\beta\sinh(\beta x_i^{(v)}) \\ & \quad - C_{i+1,1}^{(v)}\beta\cos(\beta x_i^{(v)}) + C_{i+1,2}^{(v)}\beta\sin(\beta x_i^{(v)}) - C_{i+1,3}^{(v)}\beta\cosh(\beta x_i^{(v)}) - C_{i+1,4}^{(v)}\beta\sinh(\beta x_i^{(v)}) = 0, \end{aligned} \tag{27}$$

$$\begin{aligned} & -C_{i1}^{(v)}\beta^2\sin(\beta x_i^{(v)}) - C_{i2}^{(v)}\beta^2\cos(\beta x_i^{(v)}) + C_{i3}^{(v)}\beta^2\sinh(\beta x_i^{(v)}) + C_{i4}^{(v)}\beta^2\cosh(\beta x_i^{(v)}) \\ & \quad + C_{i+1,1}^{(v)}\beta^2\sin(\beta x_i^{(v)}) + C_{i+1,2}^{(v)}\beta^2\cos(\beta x_i^{(v)}) - C_{i+1,3}^{(v)}\beta^2\sinh(\beta x_i^{(v)}) - C_{i+1,4}^{(v)}\beta^2\cosh(\beta x_i^{(v)}) = 0, \end{aligned} \tag{28}$$

$$\begin{aligned} & -C_{i1}^{(v)}\left[\beta^3\cos(\beta x_i^{(v)}) - \frac{k_{\mathrm{eff},i}^{(v)}}{EI}\sin(\beta x_i^{(v)})\right] + C_{i2}^{(v)}\left[\beta^3\sin(\beta x_i^{(v)}) - \frac{k_{\mathrm{eff},i}^{(v)}}{EI}\cos(\beta x_i^{(v)})\right] \\ & \quad + C_{i3}^{(v)}\left[\beta^3\cosh(\beta x_i^{(v)}) - \frac{k_{\mathrm{eff},i}^{(v)}}{EI}\sinh(\beta x_i^{(v)})\right] + C_{i4}^{(v)}\left[\beta^3\sinh(\beta x_i^{(v)}) - \frac{k_{\mathrm{eff},i}^{(v)}}{EI}\cosh(\beta x_i^{(v)})\right] \\ & \quad + C_{i+1,1}^{(v)}\beta^3\cos(\beta x_i^{(v)}) - C_{i+1,2}^{(v)}\beta^3\sin(\beta x_i^{(v)}) - C_{i+1,3}^{(v)}\beta^3\cosh(\beta x_i^{(v)}) - C_{i+1,4}^{(v)}\beta^3\sinh(\beta x_i^{(v)}) = 0, \end{aligned} \tag{29}$$

$$\begin{aligned} & C_{k1}^{(v)}\sin(\beta x_k^{(v)}) + C_{k2}^{(v)}\cos(\beta x_k^{(v)}) + C_{k3}^{(v)}\sinh(\beta x_k^{(v)}) + C_{k4}^{(v)}\cosh(\beta x_k^{(v)}) \\ & \quad - C_{k+1,1}^{(v)}\sin(\beta x_k^{(v)}) - C_{k+1,2}^{(v)}\cos(\beta x_k^{(v)}) - C_{k+1,3}^{(v)}\sinh(\beta x_k^{(v)}) - C_{k+1,4}^{(v)}\cosh(\beta x_k^{(v)}) = 0, \end{aligned} \tag{30}$$

$$\begin{aligned} & C_{k1}^{(v)}\beta\cos(\beta x_k^{(v)}) - C_{k2}^{(v)}\beta\sin(\beta x_k^{(v)}) + C_{k3}^{(v)}\beta\cosh(\beta x_k^{(vv)}) + C_{k4}^{(v)}\beta\sinh(\beta x_k^{(v)}) \\ & \quad - C_{k+1,1}^{(v)}\beta\cos(\beta x_k^{(v)}) + C_{k+1,2}^{(v)}\beta\sin(\beta x_k^{(v)}) - C_{k+1,3}^{(v)}\beta\cosh(\beta x_k^{(v)}) - C_{k+1,4}^{(v)}\beta\sinh(\beta x_k^{(v)}) = 0, \end{aligned} \tag{31}$$

$$\begin{aligned} & -C_{k1}^{(v)}\beta^2\sin(\beta x_k^{(v)}) - C_{k2}^{(v)}\beta^2\cos(\beta x_k^{(v)}) + C_{k3}^{(v)}\beta^2\sinh(\beta x_k^{(v)}) + C_{k4}^{(v)}\beta^2\cosh(\beta x_k^{(v)}) \\ & \quad + C_{k+1,1}^{(v)}\beta^2\sin(\beta x_k^{(v)}) + C_{k+1,2}^{(v)}\beta^2\cos(\beta x_k^{(v)}) - C_{k+1,3}^{(v)}\beta^2\sinh(\beta x_k^{(v)}) - C_{k+1,4}^{(v)}\beta^2\cosh(\beta x_k^{(v)}) = 0, \end{aligned} \tag{32}$$

$$\begin{aligned} & -C_{k1}^{(v)}\left[\beta^3\cos(\beta x_k^{(v)}) - \frac{k_{\mathrm{eff},k}^{(v)}}{EI}\sin(\beta x_k^{(v)})\right] + C_{k2}^{(v)}\left[\beta^3\sin(\beta x_k^{(v)}) - \frac{k_{\mathrm{eff},k}^{(v)}}{EI}\cos(\beta x_k^{(v)})\right] \\ & \quad + C_{k3}^{(v)}\left[\beta^3\cosh(\beta x_k^{(v)}) - \frac{k_{\mathrm{eff},k}^{(v)}}{EI}\sinh(\beta x_k^{(v)})\right] + C_{k4}^{(v)}\left[\beta^3\sinh(\beta x_k^{(v)}) - \frac{k_{\mathrm{eff},k}^{(v)}}{EI}\cosh(\beta x_k^{(v)})\right] \\ & \quad + C_{k+1,1}^{(v)}\beta^3\cos(\beta x_k^{(v)}) - C_{k+1,2}^{(v)}\beta^3\sin(\beta x_k^{(v)}) - C_{k+1,3}^{(v)}\beta^3\cosh(\beta x_k^{(v)}) - C_{k+1,4}^{(v)}\beta^3\sinh(\beta x_k^{(v)}) = 0. \end{aligned} \tag{33}$$

It is noted that, in Eqs. (25a)–(25h), the "left side" of the attaching point $i^{(v)}$ and that of the attaching point $k^{(v)}$ for the vth 2-dof spring–mass system belong to the beam segments $i^{(v)}$ and $k^{(v)}$, respectively, while the "right side" of the attaching point $i^{(v)}$ and that of the attaching point $k^{(v)}$ belong to the beam segment $i^{(v)}+1$ and $k^{(v)}+1$, respectively, thus the associated coefficients are represented by $C_{i,j}^{(v)}$, $C_{k,j}^{(v)}$, $C_{i+1,j}^{(v)}$ and $C_{k+1,j}^{(v)}$ ($j = 1, \ldots, 4$), respectively, as may be seen from Eqs. (26)–(33).

To write Eqs. (26)–(33) in matrix form gives

$$\lfloor B^{(v)} \rfloor \{C^{(v)}\} = \{0\}, \tag{34}$$

where

$$\{C^{(v)}\} = \left\{ \overline{C}_{8v-7} \quad \overline{C}_{8v-6} \quad \overline{C}_{8v-5} \quad \overline{C}_{8v-4} \quad \overline{C}_{8v-3} \quad \overline{C}_{8v-2} \quad \overline{C}_{8v-1} \quad \overline{C}_{8v} \quad \overline{C}_{8v+1} \quad \overline{C}_{8v+2} \quad \overline{C}_{8v+3} \quad \overline{C}_{8v+4} \right\} \tag{35a}$$

and

$$[B^{(v)}] = \begin{array}{c} \\ \left[\begin{array}{cccccccccccc|c}
8v\text{-}7 & 8v\text{-}6 & 8v-5 & 8v-4 & 8v-3 & 8v-2 & 8v-1 & 8v & 8v+1 & 8v+2 & 8v+3 & 8v+4 & \\
\Delta_{1v} & \Delta_{2v} & \Delta_{3v} & \Delta_{4v} & -\Delta_{1v} & -\Delta_{2v} & -\Delta_{3v} & -\Delta_{4v} & 0 & 0 & 0 & 0 & 8v-5 \\
\beta\Delta_{2v} & -\beta\Delta_{1v} & \beta\Delta_{4v} & \beta\Delta_{3v} & -\beta\Delta_{2v} & \beta\Delta_{1v} & -\beta\Delta_{4v} & -\beta\Delta_{3v} & 0 & 0 & 0 & 0 & 8v-4 \\
-\beta^2\Delta_{1v} & -\beta^2\Delta_{2v} & \beta^2\Delta_{3v} & \beta^2\Delta_{4v} & \beta^2\Delta_{1v} & \beta^2\Delta_{2v} & -\beta^2\Delta_{3v} & -\beta^2\Delta_{4v} & 0 & 0 & 0 & 0 & 8v-3 \\
-\left(\beta^3\Delta_{2v} - \frac{k^{(v)}_{\mathrm{eff},i}}{EI}\Delta_{1v}\right) & \left(\beta^3\Delta_{1v} - \frac{k^{(v)}_{\mathrm{eff},i}}{EI}\Delta_{2v}\right) & \left(\beta^3\Delta_{4v} - \frac{k^{(v)}_{\mathrm{eff},i}}{EI}\Delta_{3v}\right) & \left(\beta^3\Delta_{3v} - \frac{k^{(v)}_{\mathrm{eff},i}}{EI}\Delta_{4v}\right) & \beta^3\Delta_{2v} & -\beta^3\Delta_{1v} & -\beta^3\Delta_{4v} & -\beta^3\Delta_{3v} & 0 & 0 & 0 & 0 & 8v-2 \\
0 & 0 & 0 & 0 & \nabla_{1v} & \nabla_{2v} & \nabla_{3v} & \nabla_{4v} & -\nabla_{2v} & -\nabla_{2v} & -\nabla_{3v} & -\nabla_{4v} & 8v-1 \\
0 & 0 & 0 & 0 & \beta\nabla_{2v} & -\beta\nabla_{1v} & \beta\nabla_{4v} & \beta\nabla_{3v} & \beta\nabla_{1v} & \beta\nabla_{1v} & -\beta\nabla_{4v} & -\beta\nabla_{3v} & 8v \\
0 & 0 & 0 & 0 & -\beta^2\nabla_{1v} & -\beta^2\nabla_{2v} & \beta^2\nabla_{3v} & \beta^2\nabla_{4v} & \beta^2\nabla_{1v} & \beta^2\nabla_{2v} & -\beta^2\nabla_{3v} & -\beta^2\nabla_{4v} & 8v+1 \\
0 & 0 & 0 & 0 & -\left(\beta^3\nabla_{2v} - \frac{k^{(v)}_{\mathrm{eff},k}}{EI}\nabla_{1v}\right) & \left(\beta^3\nabla_{1v} - \frac{k^{(v)}_{\mathrm{eff},k}}{EI}\nabla_{2v}\right) & \left(\beta^3\nabla_{4v} - \frac{k^{(v)}_{\mathrm{eff},k}}{EI}\nabla_{3cv}\right) & \left(\beta^3\nabla_{3v} - \frac{k^{(v)}_{\mathrm{eff},k}}{EI}\nabla_{4v}\right) & \beta^3\nabla_{2v} & -\beta^3\nabla_{1v} & -\beta^3\nabla_{4v} & -\beta^3\nabla_{3} & 8v+2
\end{array}\right]\end{array} \tag{35b}$$

where

$$\Delta_{1v} = \sin(\beta x_i^{(v)}), \quad \Delta_{2v} = \cos(\beta x_i^{(v)}), \quad \Delta_{3v} = \sinh(\beta x_i^{(v)}), \quad \Delta_{4v} = \cosh(\beta x_i^{(v)}),$$
$$\nabla_{1v} = \sin(\beta x_k^{(v)}), \quad \nabla_{2v} = \cos(\beta x_k^{(v)}), \quad \nabla_{3v} = \sinh(\beta x_k^{(v)}), \quad \nabla_{4v} = \cosh(\beta x_k^{(v)}), \quad (v = 1 \sim n). \tag{35c}$$

5. Coefficient matrix $[B_L]$ for the left end of the beam

For a cantilever beam with left end clamped, the boundary conditions are

$$\overline{Y}(0) = 0, \quad \overline{Y}'(0) = 0. \tag{36a,b}$$

From Fig. 2 one sees that the left end of the beam, Ⓛ, coincides with the left end of the first beam segment ($i^{(1)} = 1$), hence from Eqs. (23), (36a) and (36b) one obtains

$$C_{12} + C_{14} = 0, \tag{37a}$$

$$\beta C_{11} + \beta C_{13} = 0. \tag{37b}$$

To write the last two expressions in matrix form gives

$$[B_L]\{C_L\} = \{0\}, \tag{38}$$

where

$$[B_L] = \overset{\begin{matrix}1 & 2 & 3 & 4\end{matrix}}{\begin{bmatrix} 0 & 1 & 0 & 1 \\ \beta & 0 & \beta & 0 \end{bmatrix}}\begin{matrix}1\\2\end{matrix}, \tag{39}$$

$$\{C_L\} = \left\{ \overline{C}_1 \quad \overline{C}_2 \quad \overline{C}_3 \quad \overline{C}_4 \right\}, \tag{40}$$

where the symbols [ ] and { } represent the rectangular matrix and the column vector, respectively.

In Eq. (39) and the subsequent equations, the digits shown on the top side and right side of the matrix represent the identification numbers for the dof of the associated constants $\overline{C}_i$ $(i = 1, 2, \ldots)$.

6. Coefficient matrix $[B_R]$ for the right end of the beam

For a cantilever beam with right end free, the boundary conditions are

$$\overline{Y}''(L) = 0, \quad \overline{Y}'''(L) = 0. \tag{41a,b}$$

Since the right end of the beam, Ⓡ, coincides with the right end of the $(n+1)$th beam segment (i.e., $k^{(v)} = n + 1$), as one may see from Fig. 2, from Eqs. (24), (41a) and (41b) one obtains

$$-\beta^2 \sin(\beta L)C_{n+1,1} - \beta^2 \cos(\beta L)C_{n+1,2} + \beta^2 \sinh(\beta L)C_{n+1,3} + \beta^2 \cosh(\beta L)C_{n+1,4} = 0, \tag{42a}$$

$$-\beta^3 \cos(\beta L)C_{n+1,1} + \beta^3 \sin(\beta L)C_{n+1,2} + \beta^3 \cosh(\beta L)C_{n+1,3} + \beta^3 \sinh(\beta L)C_{n+1,4} = 0. \tag{42b}$$

To write Eqs. (42a) and (42b) in matrix form gives

$$[B_R]\{C_R\} = \{0\}, \tag{43}$$

where

$$[B_R] = \overset{\begin{matrix}8n+1 & \quad 8n+2 & \quad 8n+3 & \quad 8n+4\end{matrix}}{\begin{bmatrix} -\beta^2 \sin(\beta L) & -\beta^2 \cos(\beta L) & \beta^2 \sinh(\beta L) & \beta^2 \cosh(\beta L) \\ -\beta^3 \cos(\beta L) & \beta^3 \sin(\beta L) & \beta^3 \cosh(\beta L) & \beta^3 \sinh(\beta L) \end{bmatrix}}\begin{matrix}p-1\\p\end{matrix}, \tag{44}$$

$$\{C_R\} = \left\{ \overline{C}_{8n+1} \quad \overline{C}_{8n+2} \quad \overline{C}_{8n+3} \quad \overline{C}_{8n+4} \right\}, \tag{45}$$

$$p = 8n + 4. \tag{46}$$

In the last equations, p represents the total number of equations. From the above derivations one sees that from each of the two attaching points of the 2-dof spring–mass system one may obtain four equations (including three compatibility equations and one force–equilibrium equation) and from each boundary (Ⓛ or Ⓡ) one may obtain two equations. Hence, for a beam carrying n 2-dof spring–mass systems, the total number of equations that one may obtain for the integration constants $C_{i,j}^{(v)}$, $C_{k,j}^{(v)}(v = 1, \ldots, n, \quad j = 1, \ldots, 4)$ is equal to $8n+4$, i.e., $p = 8n + 4$ as shown by Eq. (46). Of course, the total number of unknowns ($C_{i,j}^{(v)}$, $C_{k,j}^{(v)}$) is also equal to $8n+4$. From Eqs. (23) and (24) one may obtain the solutions $\overline{Y}_i^{(v)}(x_i^{(v)})$ and $\overline{Y}_k^{(v)}(x_k^{(v)})$.

7. Overall coefficient matrix $[\overline{B}]$ of the entire beam and the frequency equation

If all the unknowns $C_{i,j}^{(v)}$, $C_{k,j}^{(v)}(v = 1, \ldots, n, \quad j = 1, \ldots, 4)$ are replaced by a column vector $\{\overline{C}\}$ with coefficients $\overline{C}_l$ $(l = 1, 2, \ldots, p)$ defined by Eqs. (40), (45) and (35a), then the matrices $[B_L]$, $\lfloor B^{(v)} \rfloor$ and $[B_R]$ are similar to the element property matrices (for the FEM) with corresponding identification numbers for the dof shown on the top side and right side of the matrices defined by Eqs. (39), (44) and (35b). Basing on the assembly technique for the direct stiffness matrix method, it is easy to arrive at the following coefficient equation for the entire vibrating system

$$[B]\{\overline{C}\} = \{0\}. \tag{47}$$

Non-trivial solution of the problem requires that

$$|B| = 0, \tag{48}$$

which is the frequency equation, and the half-interval technique [17] may be used to solve the eigenvalues $\overline{\omega}_j$ $(j = 1, 2, \ldots)$. To substitute each value of $\overline{\omega}_j$ into Eq. (47) one may determine the values of unknowns $\overline{C}_l$ $(l = 1, 2, \ldots, p)$. Hence the substitution of $C_{i,j}^{(v)}$, $C_{k,j}^{(v)}(v = 1, \ldots, n, \quad j = 1, \ldots, 4)$ into Eqs. (23) and (24) will define the corresponding mode shape $\overline{Y}_i^{(v)}(x_i^{(v)})$ and $\overline{Y}_k^{(v)}(x_k^{(v)})$. For a cantilever beam carrying one $(n = 1)$ and two $(n = 2)$ 2-dof spring–mass systems, the corresponding overall coefficient matrices $[B]_{(1)}$ and $[B]_{(2)}$ are shown in Appendix A [see Eqs. (A.1) and (A.2)]. From the lengthy expressions one sees that the conventional explicit formulations are not suitable for a beam carrying more than two $(n > 2)$ 2-dof spring–mass systems. However, this is not true for the NAM adopted in this paper.

8. Coefficient matrices $[B_L]$ and $[B_R]$ for various boundary conditions

From the previous sections one finds that the form of the coefficient matrix $\lfloor B^{(v)} \rfloor$ for each attaching point of the 2-dof spring–mass system has nothing to do with the boundary conditions of the beam, hence for a "constrained" beam with various supporting conditions the only thing one should do is to modify the values of the two boundary matrices $[B_L]$ and $[B_R]$ defined by Eqs. (39) and (44), respectively, according to the actual boundary conditions. And then the same numerical assembly procedures introduced in the last section may be followed. This is one of the predominant advantages of the NAM. The boundary matrices $[B_L]$ and $[B_R]$ for various boundary conditions are placed in Appendix B at the end of this paper.

9. Numerical results and discussions

The material properties and dimensions of the beam studied in this paper are: beam length $L = 4\,\mathrm{m}$, Young's modulus $E = 2.0 \times 10^{11}\,\mathrm{N/m^2}$, cross-sectional area $A = 0.15\,\mathrm{m^2}$, moment of inertia of the cross-sectional area $I = 3.125 \times 10^{-3}\,\mathrm{m^4}$ and mass density of the beam material $\rho = 7860\,\mathrm{kg/m^3}$.

For convenience, the two-letter acronyms, CF, CS, CC and SS, are used to denote the clamped–free (CF), clamped–simply supported (CS), clamped–clamped (CC), and simply supported–simply supported (SS) boundary conditions of the beam, respectively.

Table 1
The lowest five natural frequencies $\overline{\omega}_j$ ($j = 1, \ldots, 5$) for a uniform beam carrying a 2-dof spring–mass system at $x_i^{(1)} = 1.4$ m and $x_k^{(1)} = 2.6$ m with $a_1^{(1)} = 0.6$ m and $a_2^{(1)} = 0.6$ m

Boundary conditions	Methods	Natural frequencies (rad/s)				
		$\overline{\omega}_1$	$\overline{\omega}_2$	$\overline{\omega}_3$	$\overline{\omega}_4$	$\overline{\omega}_5$
SS	NAM	434.81	1747.30	4037.45	6768.48	9657.59
	FEM	434.70	1747.70	4037.54	6773.82	9633.12
	Ref. [14]	434	1747	4037	6768	9657
CC	NAM	983.47	2694.21	5499.06	8438.83	9949.97
	FEM	983.20	2695.18	5499.38	8449.96	9930.93
	Ref. [14]	983	2694	5499	8438	9949

NAM = numerical assembly method; FEM = finite element method.

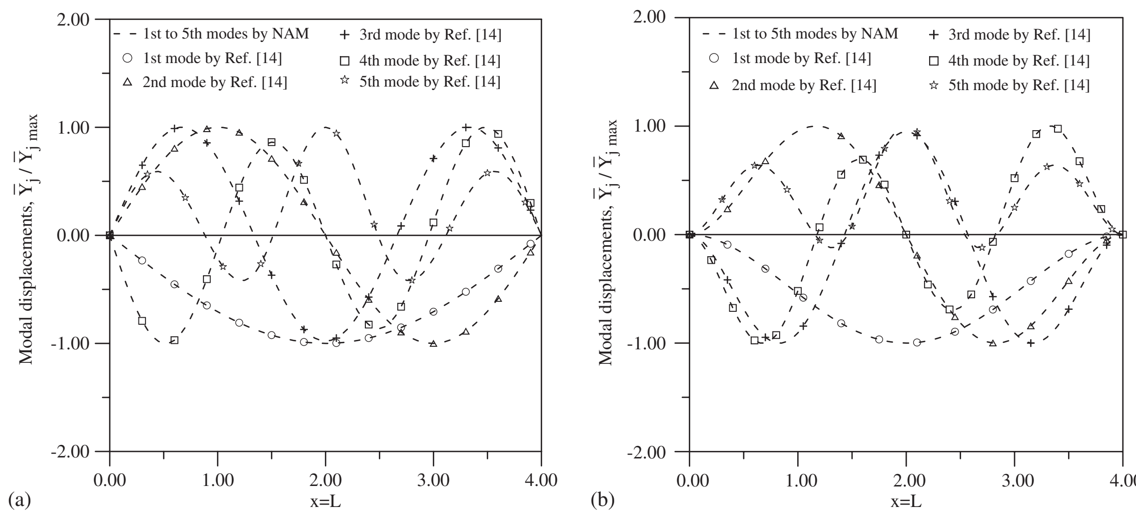

Fig. 3. The lowest five mode shapes $\overline{Y}_j(x)$ ($j = 1,\ldots,5$) for a uniform beam carrying a 2-dof spring–mass system obtained from the NAM and those from Chang et al. [14] with the support conditions: (a) SS and (b) CC.

9.1. Comparison with the existing results

To check the reliability of the theory presented and the computer programs developed for this paper, a uniform beam (SS and CC) carrying a 2-dof spring–mass system located at $x_i^{(1)} = 1.4$ m and $x_k^{(1)} = 2.6$ m with $a_1^{(1)} = 0.6$ m and $a_2^{(1)} = 0.6$ m presented by Chang et al. [14] is studied here. In addition to the material properties and dimensions for the beam given above, the other given data for the 2-dof spring–mass system are: $m_e^{(1)} = 200$ kg, $J_e^{(1)} = 70.8333$ kg m^2, $k_1^{(1)} = k_2^{(1)} = 1 \times 10^{10}$ N/m.

Table 1 shows the lowest five natural frequencies $\overline{\omega}_j$ ($j = 1, \ldots, 5$) and Fig. 3 shows the five corresponding mode shapes $\overline{Y}_j$ ($j = 1, \ldots, 5$). From Table 1 and Fig. 3, we can see that the lowest five natural frequencies and the corresponding mode shapes obtained from the present method (NAM) are in good agreement with those obtained from Chang et al. [14] or the FEM.

Table 2
The lowest five natural frequencies $\overline{\omega}_j$ ($j = 1, \ldots, 5$) for a uniform beam carrying a 2-dof spring–mass system (with $x_i^{(1)} = 1.4\,\text{m}$, $x_k^{(1)} = 2.6\,\text{m}$, $a_1^{(1)} = 0.6\,\text{m}$, $a_2^{(1)} = 0.6\,\text{m}$, $k_1^{(1)} = k_2^{(1)} = 1 \times 10^{10}\,\text{N/m}$, $m_e^{(1)} = 200\,\text{kg}$, $J_e^{(1)} = 70.8333\,\text{kg}\,\text{m}^2$)

Boundary conditions	Methods	Natural frequencies (rad/s)				
		$\overline{\omega}_1$	$\overline{\omega}_2$	$\overline{\omega}_3$	$\overline{\omega}_4$	$\overline{\omega}_5$
CF	NAM	157.96	978.39	2704.52	5499.75	8431.55
		(0.0066%)	(0.0183%)	(−0.0328%)	(−0.0058%)	(−0.1327%)
	FEM	157.95	978.21	2705.41	5500.08	8442.76
CS	NAM	678.96	2202.13	4706.90	7654.52	9861.08
		(0.0245%)	(−0.0243%)	(−0.0121%)	(−0.0832%)	(0.2318%)
	FEM	678.79	2202.66	4707.47	7660.90	9838.22
SS	NAM	434.81	1747.30	4037.45	6768.48	9657.59
		(0.0255%)	(−0.0234%)	(−0.0023%)	(−0.0789%)	(0.2533%)
	FEM	434.70	1747.71	4037.54	6773.82	9633.12
CC	NAM	983.47	2694.21	5499.06	8438.83	9949.97
		(0.0273%)	(−0.0359%)	(−0.0057%)	(−0.1318%)	(0.1913%)
	FEM	983.20	2695.18	5499.37	8449.95	9930.93

Note: The percentage differences between $\overline{\omega}_{j\text{NAM}}$ and $\overline{\omega}_{j\text{FEM}}$ shown in the parentheses ( ) are determined with the formula: $\varepsilon_j^* = (\overline{\omega}_{j\text{NAM}} - \overline{\omega}_{j\text{FEM}}) \times 100\%/\overline{\omega}_{j\text{NAM}}$.

9.2. *Free vibration analysis of the beam*

9.2.1. *Case 1: carrying a 2-dof spring–mass system*

For a uniform beam carrying a 2-dof spring–mass system located at $x_i^{(1)} = 1.4\,\text{m}$ and $x_k^{(1)} = 2.6\,\text{m}$ with $a_1^{(1)} = 0.6\,\text{m}$ and $a_2^{(1)} = 0.6\,\text{m}$ and the physical properties of the 2-dof spring–mass system given by $m_e^{(1)} = 200\,\text{kg}$, $J_e^{(1)} = 70.8333\,\text{kg}\,\text{m}^2$ and $k_1^{(1)} = k_2^{(1)} = 1 \times 10^{10}\,\text{N/m}$, the calculated lowest five natural frequencies $\overline{\omega}_j$ ($j = 1, \ldots, 5$) are shown in Table 2 and the corresponding mode shapes $\overline{Y}_j$ ($j = 1, \ldots, 5$) for the four types of boundary conditions are shown in Figs. 4(a)–(d), respectively. It can be seen that the lowest five natural frequencies of the "constrained" beam, $\overline{\omega}_j$ ($j = 1, \ldots, 5$), shown in Table 2 are smaller than the corresponding ones of the "bare" beam, ω_j ($j = 1, \ldots, 5$), shown in Table 3. The difference between $\overline{\omega}_j$ and ω_j, $\Delta\omega_j = \omega_j - \overline{\omega}_j$, increases with increasing the mode number j. But the lowest five mode shapes of the "constrained" beam shown in Fig. 4 look like those of the "bare" beam shown in Fig. 5.

The percentage differences between $\overline{\omega}_{j\text{NAM}}$ and $\overline{\omega}_{j\text{FEM}}$ shown in the parentheses ( ) of Table 2 are calculated with the formula: $\varepsilon_j^* = (\overline{\omega}_{j\text{NAM}} - \overline{\omega}_{j\text{FEM}}) \times 100\%/\overline{\omega}_{j\text{NAM}}$, where $\overline{\omega}_{j\text{NAM}}$ and $\overline{\omega}_{j\text{FEM}}$ denote the jth natural frequencies of the "constrained" beam obtained from the NAM and the FEM, respectively. From Table 2 one finds that the maximum value of ε_j^* is $\varepsilon_5^* = 0.2533\%$ (for the SS boundary condition), hence the accuracy of the NAM is good.

From Fig. 4 one sees that the mode shapes of the beam are symmetrical for the SS or CC beam. This is reasonable because the symmetrical beam carries a 2-dof spring–mass system at the symmetrical location.

9.2.2. *Case 2: carrying five 2-dof spring–mass systems*

This subsection studied a uniform beam carrying five 2-dof spring–mass systems with locations and magnitudes given in Table 4. For the four types of boundary conditions (CF, CS, SS and CC), the lowest five natural frequencies $\overline{\omega}_j$ ($j = 1, \ldots, 5$) are shown in Table 5 and the corresponding mode shapes $\overline{Y}_j$ ($j = 1, \ldots, 5$) are shown in Figs. 6(a)–(d), respectively. The maximum value of the percentage difference is found to be $\varepsilon_5^* = 0.2438\%$ (for the CS boundary condition), hence the accuracy of the NAM is good.

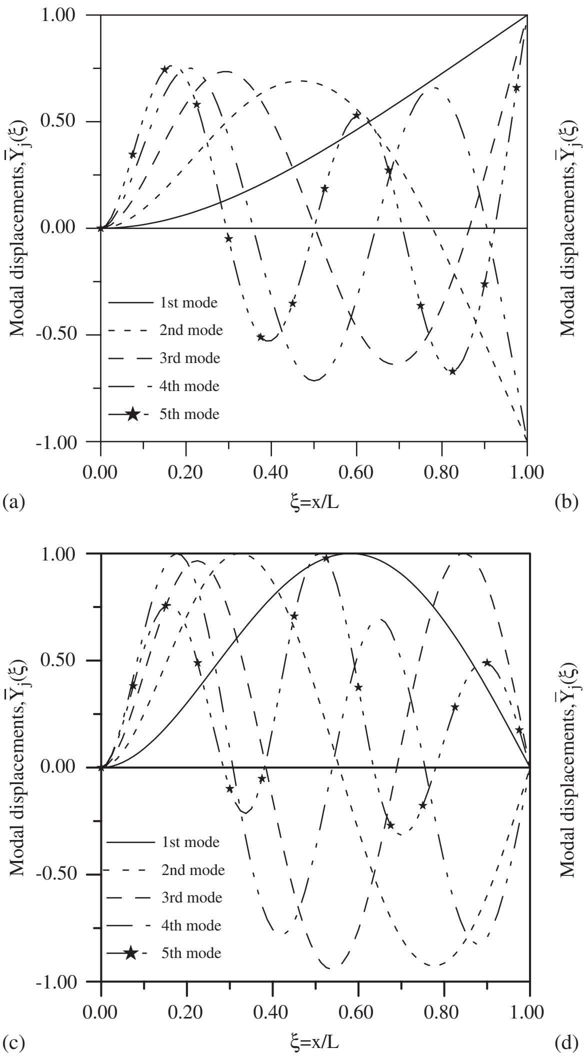

Fig. 4. The lowest five mode shapes $\overline{Y}_j$ ($j = 1,\ldots,5$) for a uniform beam carrying a 2-dof spring–mass system ($x_i^{(1)} = 1.4\,\mathrm{m}$, $x_k^{(1)} = 2.6\,\mathrm{m}$, $a_1^{(1)} = 0.6\,\mathrm{m}$, $a_2^{(1)} = 0.6\,\mathrm{m}$, $k_1^{(1)} = k_2^{(1)} = 1\times10^{10}\,\mathrm{N/m}$, $m_e^{(1)} = 200\,\mathrm{kg}$, $J_e^{(1)} = 70.8333\,\mathrm{kg\,m^2}$) with the support conditions: (a) CF, (b) CS, (c) SS and (d) CC.

Table 3
The lowest five natural frequencies ω_j ($j = 1,\ldots,5$) for the "bare" beam (without carrying any 2-dof spring–mass systems)

Boundary condition	Methods	Natural frequencies (rad/s)				
		ω_1	ω_2	ω_3	ω_4	ω_5
CF	NAM	159.99	1002.69	2807.56	5501.69	9094.69
	FEM	159.99	1002.69	2807.60	5502.04	9096.23
CS	NAM	701.61	2273.67	4743.83	8112.24	12378.88
	FEM	701.61	2273.70	4744.06	8113.34	12382.79
SS	NAM	449.12	1796.48	4042.08	7185.93	11228.01
	FEM	449.12	1796.49	4042.22	7186.70	11230.93
CC	NAM	1018.11	2806.45	5501.76	9094.69	13585.90
	FEM	1018.11	2806.49	5502.11	9096.25	13591.05

NAM = numerical assembly method; FEM = finite element method.

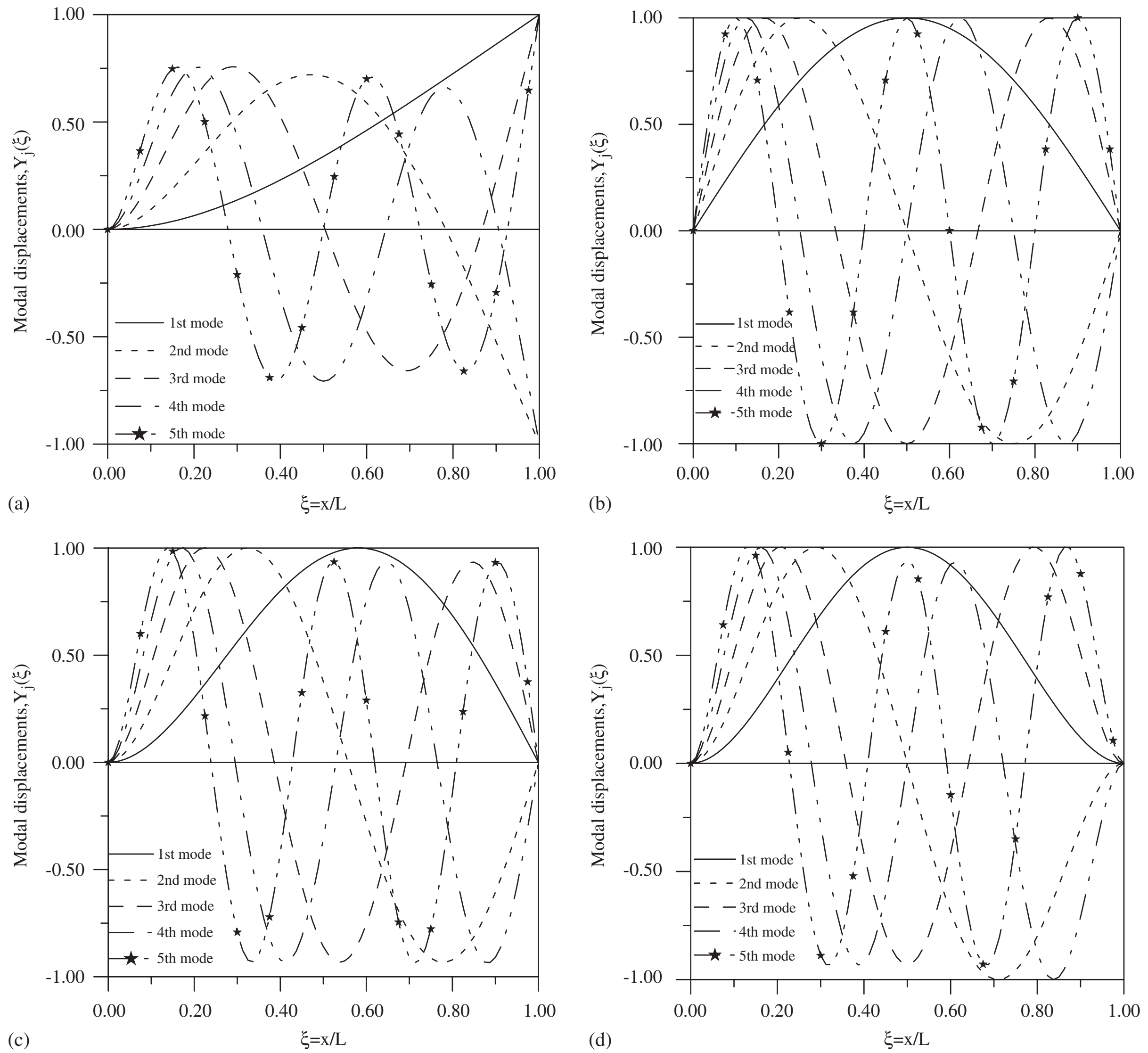

Fig. 5. The lowest five mode shapes Y_j ($j = 1, \ldots, 5$) for the "bare" beam (without carrying any 2-dof spring–mass systems) with the support conditions: (a) CF, (b) CS, (c) SS and (d) CC.

Table 4
The locations and physical properties of the five 2-dof spring–mass systems attached to the uniform beam

Numbering of systems (v)	Locations		Physical properties of the 2-dof spring-mass systems					
	$x_i^{(v)}$ (m)	$x_k^{(v)}$ (m)	$a_1^{(v)}$ (m)	$a_2^{(v)}$ (m)	$k_1^{(v)}$ (N/m)	$k_2^{(v)}$ (N/m)	$m_e^{(v)}$ (kg)	$J_e^{(v)}$ (kg m^2)
1	0.3	0.9	0.3	0.3	1×10^{10}	1×10^{10}	200	70.8333
2	1.0	1.6	0.3	0.3	1×10^{10}	1×10^{10}	300	106.2499
3	1.7	2.3	0.3	0.3	1×10^{10}	1×10^{10}	400	141.6666
4	2.4	3.0	0.3	0.3	1×10^{10}	1×10^{10}	300	106.2499
5	3.1	3.7	0.3	0.3	1×10^{10}	1×10^{10}	200	70.8333

Table 5
The lowest five natural frequencies $\overline{\omega}_j$ ($j = 1, \ldots, 5$) for a uniform beam carrying five 2-dof spring–mass systems with parameters as shown in Table 4

Boundary condition	Methods	Natural frequencies (rad/s)				
		$\overline{\omega}_1$	$\overline{\omega}_2$	$\overline{\omega}_3$	$\overline{\omega}_4$	$\overline{\omega}_5$
CF	NAM	126.56 (0.0085%)	749.33 (0.0193%)	2028.31 (−0.0228%)	3283.11 (0.0098%)	3511.91 (0.1527%)
	FEM	126.55	747.89	2032.94	3279.89	3506.54
CS	NAM	501.25 (0.0325%)	1638.15 (0.0273%)	2998.00 (−0.0181%)	3472.02 (−0.0930%)	3850.07 (0.2438%)
	FEM	501.09	1637.71	2998.54	3475.25	3840.69
SS	NAM	322.28 (0.0215%)	1328.89 (−0.0258%)	2718.96 (0.0020%)	3403.68 (−0.0851%)	3840.49 (0.2395%)
	FEM	322.21	1329.24	2718.90	3406.58	3831.30
CC	NAM	716.71 (0.0281%)	1965.14 (−0.0391%)	3237.42 (−0.0081%)	3499.94 (0.1257%)	3865.02 (0.2158%)
	FEM	716.51	1965.91	3237.68	3495.54	3856.68

Note: The percentage differences between $\overline{\omega}_{j\mathrm{NAM}}$ and $\overline{\omega}_{j\mathrm{FEM}}$ shown in the parentheses ( ) are determined with the formula: $\varepsilon_j^* = (\overline{\omega}_{j\mathrm{NAM}} - \overline{\omega}_{j\mathrm{FEM}}) \times 100\%/\overline{\omega}_{j\mathrm{NAM}}$.

The symmetrical mode shapes shown in Figs. 6(c) and (d) are due to the symmetric distribution of locations and magnitudes for five 2-dof spring–mass systems along the beam length. Comparing Figs. 6(b)–(d) (for constrained beam) with Figs. 5(b)–(d) (for bare beam), one sees that the major differences are in the third, fourth and fifth mode shapes. The amplitudes for the middle points of the third and fourth mode shapes shown in Figs. 6(b)–(d) are smaller than those for the third and fourth mode shapes shown in Figs. 5(b)–(d). Besides, the node number for each of the fifth mode shapes shown in Figs. 6(b)–(d) is 2, this is different from that shown in Figs. 5(b)–(d) with the node number for each of the fifth mode shapes being 4. Where the "node number" for a mode shape refers to the number of "intersecting points" between the mode shape and the horizontal line representing the static equilibrium position of the beam. The main reason for the above-mentioned phenomena is that the magnitudes of the lumped mass (m_e) and mass moment of inertia (J_e) for the 2-dof spring–mass system located at the middle part of the "constrained" beam are larger than those for the other four 2-dof spring–mass systems.

9.3. *Influence of magnitudes of lumped mass m_e and mass moment of inertia J_e*

Theoretically, we can reduce the magnitude of lumped mass (m_e) of a 2-dof spring–mass system attached to a cantilever (CF) beam to raise the lowest two natural frequencies of the constrained beam. Similarly, this is also true for the mass moment of inertia (J_e) of the 2-dof spring–mass system. The influence of magnitude of the lumped mass m_e of the 2-dof spring–mass system on the lowest two natural frequencies of the constrained CF beam ($J_e = 70.8333\,\mathrm{kg\,m^2}$ and $708.333\,\mathrm{kg\,m^2}$ with $x_i^{(1)} = 1.4\,\mathrm{m}$, $x_k^{(1)} = 2.6\,\mathrm{m}$, $a_1^{(1)} = 0.6\,\mathrm{m}$, and $a_2^{(1)} = 0.6\,\mathrm{m}$) is shown in Fig. 7(a) for the first frequency $\overline{\omega}_1$ and in Fig. 7(b) for the second frequency $\overline{\omega}_2$. Similarly, the influence of magnitude of the mass moment of inertia (J_e) of the 2-dof spring–mass system on the lowest two natural frequencies of the CF beam ($m_e = 200$ and $2000\,\mathrm{kg}$ with $x_i^{(1)} = 1.4\,\mathrm{m}$, $x_k^{(1)} = 2.6\,\mathrm{m}$, $a_1^{(1)} = 0.6\,\mathrm{m}$, and $a_2^{(1)} = 0.6\,\mathrm{m}$) is shown in Figs. 8(a) and (b) for the first frequency $\overline{\omega}_1$ and second frequency $\overline{\omega}_2$, respectively. It is evident that the lowest two natural frequencies of the CF beam decrease with the increase of the magnitude of lumped mass (or mass moment of inertia) if the magnitude of mass moment of inertia (or lumped mass) of the 2-dof system is kept unchanged.

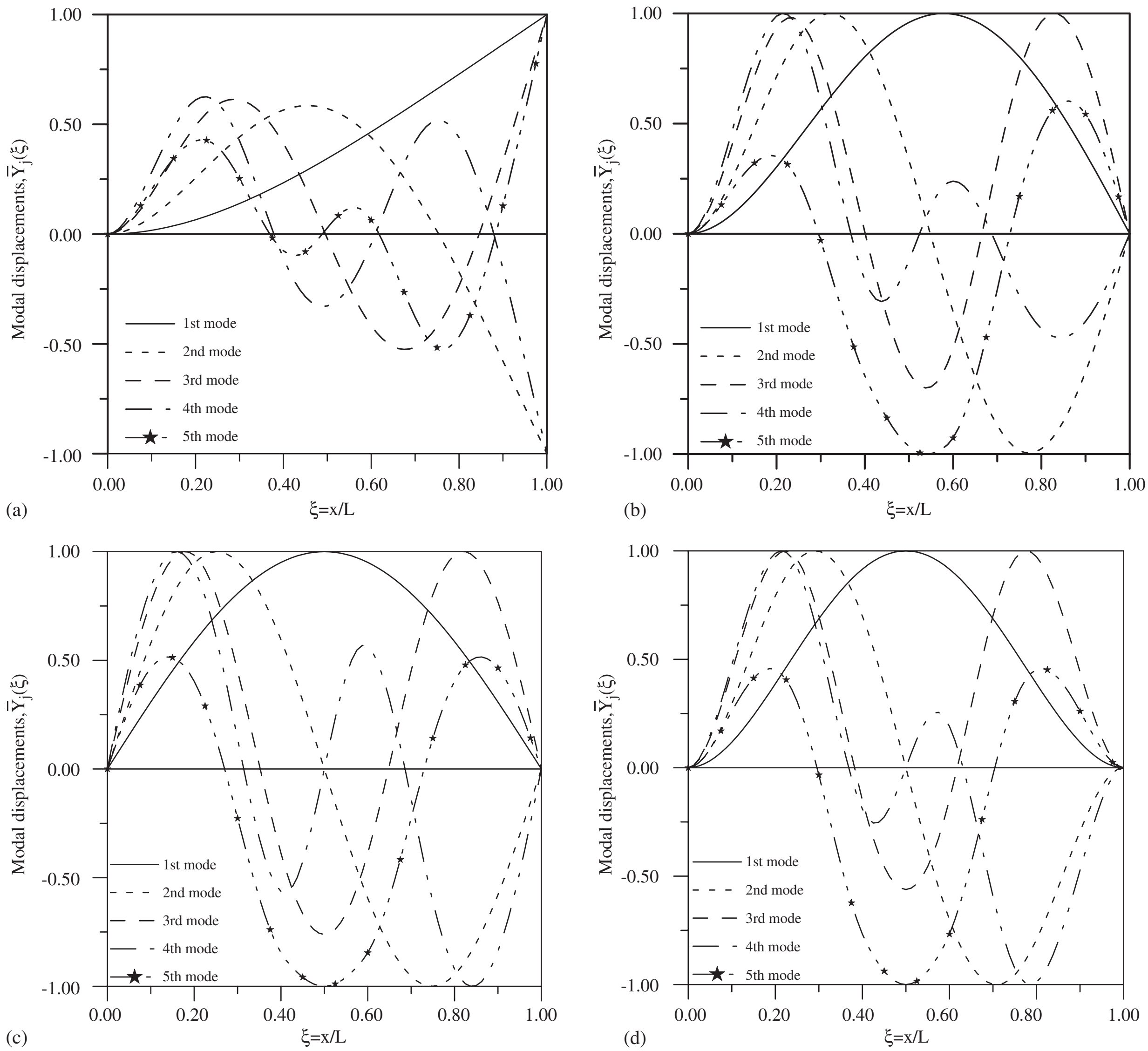

Fig. 6. The lowest five mode shapes $\overline{Y}_j$ ($j = 1, \ldots, 5$) of a uniform beam carrying five 2-dof spring–mass systems with locations and magnitudes as shown in Table 4 for the support conditions: (a) CF, (b) CS, (c) SS and (d) CC.

10. Conclusions

(1) For a uniform beam with various boundary conditions and carrying any number of two-degree-of-freedom (2-dof) spring–mass systems, one may determine its exact natural frequencies and the corresponding mode shapes using the NAM with no difficulty.
(2) For a uniform beam carrying multiple 2-dof spring–mass systems (or attachments), its natural frequencies and mode shapes are significantly influenced by the magnitude and location for each of the attachments along the beam length.
(3) Since the formulation of the NAM is based on the continuous model, the solutions achieved by using NAM are the "exact" ones, but this is not true for the solutions obtained from the conventional FEM.

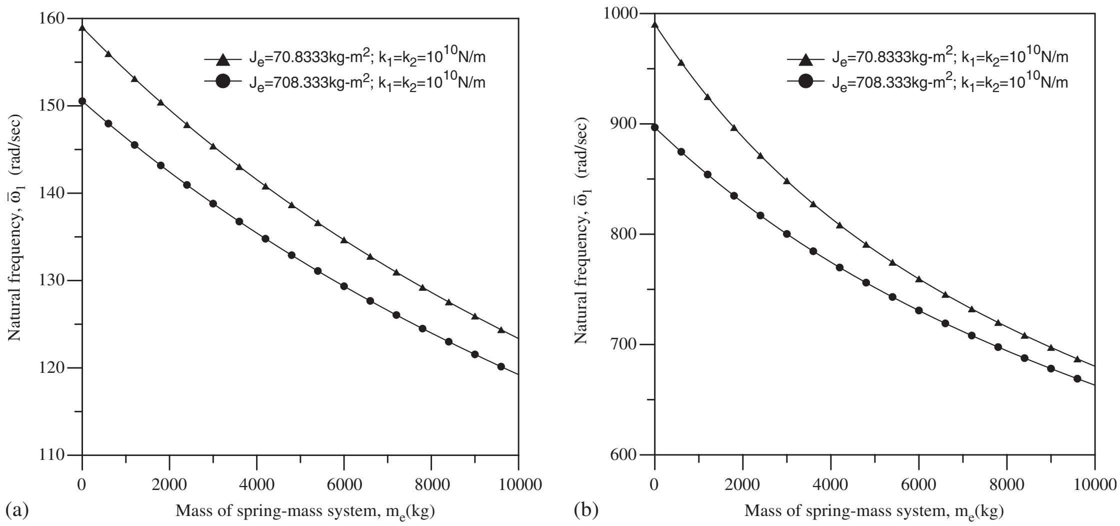

Fig. 7. Influence of magnitude of the lumped mass m_e of the 2-dof spring–mass system on the lowest two natural frequencies of the constrained CF beam ($J_e = 70.8333\,\text{kg}\,\text{m}^2$ and $708.333\,\text{kg}\,\text{m}^2$ with $x_i^{(1)} = 1.4\,\text{m}$, $x_k^{(1)} = 2.6\,\text{m}$, $a_1^{(1)} = 0.6\,\text{m}$, and $a_2^{(1)} = 0.6\,\text{m}$): (a) first frequency $\overline{\omega}_1$; (b) second frequency $\overline{\omega}_2$.

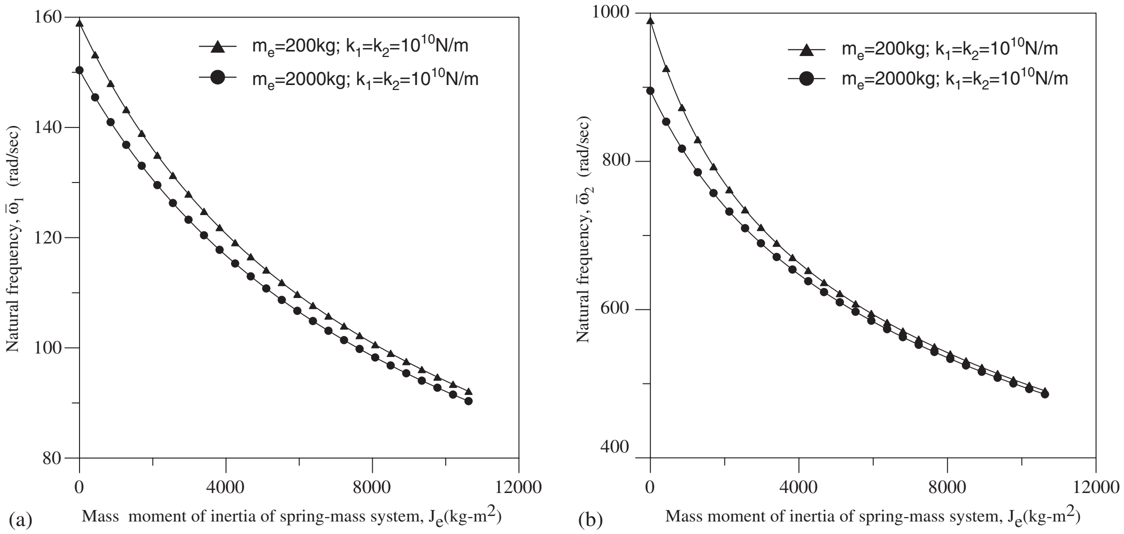

Fig. 8. Influence of magnitude of the mass moment of inertia J_e of the 2-dof spring–mass system on the lowest two natural frequencies of the constrained CF beam ($m_e = 200\,\text{kg}$ and $2000\,\text{kg}$ with $x_i^{(1)} = 1.4\,\text{m}$, $x_k^{(1)} = 2.6\,\text{m}$, $a_1^{(1)} = 0.6\,\text{m}$, and $a_2^{(1)} = 0.6\,\text{m}$): (a) first frequency $\overline{\omega}_1$; (b) second frequency $\overline{\omega}_2$.

Appendix A

For a uniform cantilever (CF) beam respectively carrying one and two 2-dof spring–mass systems, the "explicit" expressions for the overall coefficient matrices $[B]_{(1)}$ and $[B]_{(2)}$ were given by Eqs. (A.1) and (A.2), respectively.

$$
[B]_{(1)} = \begin{array}{c} \\ \end{array}
\begin{bmatrix}
\overline{C}_1 & \overline{C}_2 & \overline{C}_3 & \overline{C}_4 & \overline{C}_5 & \overline{C}_6 & \overline{C}_7 & \overline{C}_8 & \overline{C}_9 & \overline{C}_{10} & \overline{C}_{11} & \overline{C}_{12} & \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\beta & 0 & \beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
\Delta_{11} & \Delta_{21} & \Delta_{31} & \Delta_{41} & -\Delta_{11} & -\Delta_{21} & -\Delta_{31} & -\Delta_{41} & 0 & 0 & 0 & 0 & 3 \\
\beta\Delta_{21} & -\beta\Delta_{11} & \beta\Delta_{41} & \beta\Delta_{31} & -\beta\Delta_{21} & \beta\Delta_{11} & -\beta\Delta_{41} & -\beta\Delta_{31} & 0 & 0 & 0 & 0 & 4 \\
-\beta^2\Delta_{11} & -\beta^2\Delta_{21} & \beta^2\Delta_{31} & \beta^2\Delta_{41} & \beta^2\Delta_{11} & \beta^2\Delta_{21} & -\beta^2\Delta_{31} & -\beta^2\Delta_{41} & 0 & 0 & 0 & 0 & 5 \\
\Re_{11} & \Re_{21} & \Re_{31} & \Re_{41} & \beta^3\Delta_{21} & -\beta^3\Delta_{11} & -\beta^3\Delta_{41} & -\beta^3\Delta_{31} & 0 & 0 & 0 & 0 & 6 \\
0 & 0 & 0 & 0 & \nabla_{11} & \nabla_{21} & \nabla_{31} & \nabla_{41} & -\nabla_{11} & -\nabla_{21} & -\nabla_{31} & -\nabla_{41} & 7 \\
0 & 0 & 0 & 0 & \beta\nabla_{21} & -\beta\nabla_{11} & \beta\nabla_{41} & \beta\nabla_{31} & -\beta\nabla_{21} & \beta\nabla_{11} & -\beta\nabla_{41} & -\beta\nabla_{31} & 8 \\
0 & 0 & 0 & 0 & -\beta^2\nabla_{11} & -\beta^2\nabla_{21} & \beta^2\nabla_{31} & \beta^2\nabla_{41} & \beta^2\nabla_{11} & \beta^2\nabla_{21} & -\beta^2\nabla_{31} & -\beta^2\nabla_{41} & 9 \\
0 & 0 & 0 & 0 & \tilde{\Re}_{11} & \tilde{\Re}_{21} & \tilde{\Re}_{31} & \tilde{\Re}_{41} & \beta^3\nabla_{21} & -\beta^3\nabla_{11} & -\beta^3\nabla_{41} & -\beta^3\nabla_{31} & 10 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\beta^2\sin(\beta L) & -\beta^2\cos(\beta L) & \beta^2\sinh(\beta L) & \beta^2\cosh(\beta L) & 11 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\beta^3\cos(\beta L) & \beta^3\sin(\beta L) & \beta^3\cosh(\beta L) & \beta^3\sinh(\beta L) & 12
\end{bmatrix}
\tag{A.1}
$$

where

$$
\Delta_{11} = \sin(\beta x_i^{(1)}),\quad \Delta_{21} = \cos(\beta x_i^{(1)}),\quad \Delta_{31} = \sinh(\beta x_i^{(1)}),\quad \Delta_{41} = \cosh(\beta x_i^{(1)}),\quad \nabla_{11} = \sin(\beta x_k^{(1)}),\quad \nabla_{21} = \cos(\beta x_k^{(1)})
$$

$$
\nabla_{31} = \sinh(\beta x_k^{(1)}),\quad \nabla_{41} = \cosh(\beta x_k^{(1)}),\quad \Re_{11} = -\left(\beta^3\Delta_{21} - \frac{k_{\mathrm{eff},i}^{(1)}}{EI}\Delta_{11}\right),\quad \Re_{21} = \left(\beta^3\Delta_{11} - \frac{k_{\mathrm{eff},i}^{(1)}}{EI}\Delta_{21}\right),\quad \Re_{31} = \left(\beta^3\Delta_{41} - \frac{k_{\mathrm{eff},i}^{(1)}}{EI}\Delta_{31}\right),\quad \Re_{41} = \left(\beta^3\Delta_{31} - \frac{k_{\mathrm{eff},i}^{(1)}}{EI}\Delta_{41}\right)
$$

$$
\tilde{\Re}_{11} = -\left(\beta^3\nabla_{21} - \frac{k_{\mathrm{eff},k}^{(1)}}{EI}\nabla_{11}\right),\quad \tilde{\Re}_{21} = \left(\beta^3\nabla_{11} - \frac{k_{\mathrm{eff},k}^{(1)}}{EI}\nabla_{21}\right),\quad \tilde{\Re}_{31} = \left(\beta^3\nabla_{41} - \frac{k_{\mathrm{eff},k}^{(1)}}{EI}\nabla_{31}\right),\quad \tilde{\Re}_{41} = \left(\beta^3\nabla_{31} - \frac{k_{\mathrm{eff},k}^{(1)}}{EI}\nabla_{41}\right).
$$

$$
[B]_{(2)} = \begin{array}{cccccccccccccccccccc|c}
\overline{C}_1 & \overline{C}_2 & \overline{C}_3 & \overline{C}_4 & \overline{C}_5 & \overline{C}_6 & \overline{C}_7 & \overline{C}_8 & \overline{C}_9 & \overline{C}_{10} & \overline{C}_{11} & \overline{C}_{12} & \overline{C}_{13} & \overline{C}_{14} & \overline{C}_{15} & \overline{C}_{16} & \overline{C}_{17} & \overline{C}_{18} & \overline{C}_{19} & \overline{C}_{20} & \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\beta & 0 & \beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
\Delta_{11} & \Delta_{21} & \Delta_{31} & \Delta_{41} & -\Delta_{11} & -\Delta_{21} & -\Delta_{31} & -\Delta_{41} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\
\beta\Delta_{21} & -\beta\Delta_{11} & \beta\Delta_{41} & \beta\Delta_{31} & -\beta\Delta_{21} & \beta\Delta_{11} & -\beta\Delta_{41} & -\beta\Delta_{31} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\
-\beta^2\Delta_{11} & -\beta^2\Delta_{21} & \beta^2\Delta_{31} & \beta^2\Delta_{41} & \beta^2\Delta_{11} & \beta^2\Delta_{21} & -\beta^2\Delta_{31} & -\beta^2\Delta_{41} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \\
\Re_{11} & \Re_{21} & \Re_{31} & \Re_{41} & \beta^3\Delta_{21} & -\beta^3\Delta_{11} & -\beta^3\Delta_{41} & -\beta^3\Delta_{31} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\
0 & 0 & 0 & 0 & \nabla_{11} & \nabla_{21} & \nabla_{31} & \nabla_{41} & -\nabla_{11} & -\nabla_{21} & -\nabla_{31} & -\nabla_{41} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 \\
0 & 0 & 0 & 0 & \beta\nabla_{21} & -\beta\nabla_{11} & \beta\nabla_{41} & \beta\nabla_{31} & -\beta\nabla_{21} & \beta\nabla_{11} & -\beta\nabla_{41} & -\beta\nabla_{31} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 \\
0 & 0 & 0 & 0 & -\beta^2\nabla_{11} & -\beta^2\nabla_{21} & \beta^2\nabla_{31} & \beta^2\nabla_{41} & \beta^2\nabla_{11} & \beta^2\nabla_{21} & -\beta^2\nabla_{31} & -\beta^2\nabla_{41} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 \\
0 & 0 & 0 & 0 & \tilde{\Re}_{11} & \tilde{\Re}_{21} & \tilde{\Re}_{31} & \tilde{\Re}_{41} & \beta^3\nabla_{21} & -\beta^3\nabla_{11} & -\beta^3\nabla_{41} & -\beta^3\nabla_{31} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Delta_{12} & \Delta_{22} & \Delta_{32} & \Delta_{42} & -\Delta_{12} & -\Delta_{22} & -\Delta_{32} & -\Delta_{42} & 0 & 0 & 0 & 0 & 11 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta\Delta_{22} & -\beta\Delta_{12} & \beta\Delta_{42} & \beta\Delta_{32} & -\beta\Delta_{22} & \beta\Delta_{12} & -\beta\Delta_{42} & -\beta\Delta_{32} & 0 & 0 & 0 & 0 & 12 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\beta^2\Delta_{12} & -\beta^2\Delta_{22} & \beta^2\Delta_{32} & \beta^2\Delta_{42} & \beta^2\Delta_{12} & \beta^2\Delta_{22} & -\beta^2\Delta_{32} & -\beta^2\Delta_{42} & 0 & 0 & 0 & 0 & 13 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Re_{12} & \Re_{22} & \Re_{32} & \Re_{42} & \beta^3\Delta_{22} & -\beta^3\Delta_{12} & -\beta^3\Delta_{42} & -\beta^3\Delta_{32} & 0 & 0 & 0 & 0 & 14 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \nabla_{12} & \nabla_{22} & \nabla_{32} & \nabla_{42} & -\nabla_{12} & -\nabla_{22} & -\nabla_{32} & -\nabla_{42} & 15 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta\nabla_{22} & -\beta\nabla_{12} & \beta\nabla_{42} & \beta\nabla_{32} & -\beta\nabla_{22} & \beta\nabla_{12} & -\beta\nabla_{42} & -\beta\nabla_{32} & 16 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\beta^2\nabla_{12} & -\beta^2\nabla_{22} & \beta^2\nabla_{32} & \beta^2\nabla_{42} & \beta^2\nabla_{12} & \beta^2\nabla_{22} & -\beta^2\nabla_{32} & -\beta^2\nabla_{42} & 17 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{\Re}_{12} & \tilde{\Re}_{22} & \tilde{\Re}_{32} & \tilde{\Re}_{42} & \beta^3\nabla_{22} & -\beta^3\nabla_{12} & -\beta^3\nabla_{42} & -\beta^3\nabla_{32} & 18 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\beta^2\sin(\beta L) & -\beta^2\cos(\beta L) & \beta^2\sinh(\beta L) & \beta^2\cosh(\beta L) & 19 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\beta^3\cos(\beta L) & \beta^3\sin(\beta L) & \beta^3\cosh(\beta L) & \beta^3\sinh(\beta L) & 20
\end{array}, \tag{A.2}
$$

where

$$\Delta_{1v} = \sin(\beta x_i^{(v)}), \quad \Delta_{2v} = \cos(\beta x_i^{(v)}), \quad \Delta_{3v} = \sinh(\beta x_i^{(v)}), \quad \Delta_{4v} = \cosh(\beta x_i^{(v)}), \quad \nabla_{1v} = \sin(\beta x_k^{(v)}),$$

$$\nabla_{2v} = \cos(\beta x_k^{(v)}), \quad \nabla_{3v} = \sinh(\beta x_k^{(v)}), \quad \nabla_{4v} = \cosh(\beta x_k^{(v)}), \quad \Re_{1v} = -\left(\beta^3 \Delta_{2v} - \frac{k_{\mathrm{eff},i}^{(v)}}{EI}\Delta_{1v}\right),$$

$$\Re_{2v} = \left(\beta^3 \Delta_{1v} - \frac{k_{\mathrm{eff},i}^{(v)}}{EI}\Delta_{2v}\right), \quad \Re_{3v} = \left(\beta^3 \Delta_{4v} - \frac{k_{\mathrm{eff},i}^{(v)}}{EI}\Delta_{3v}\right), \quad \Re_{4v} = \left(\beta^3 \Delta_{3v} - \frac{k_{\mathrm{eff},i}^{(v)}}{EI}\Delta_{4v}\right),$$

$$\tilde{\Re}_{1v} = -\left(\beta^3 \nabla_{2v} - \frac{k_{\mathrm{eff},k}^{(v)}}{EI}\nabla_{1v}\right), \quad \tilde{\Re}_{2v} = \left(\beta^3 \nabla_{1v} - \frac{k_{\mathrm{eff},k}^{(v)}}{EI}\nabla_{2v}\right), \quad \tilde{\Re}_{3v} = \left(\beta^3 \nabla_{4v} - \frac{k_{\mathrm{eff},k}^{(v)}}{EI}\nabla_{3v}\right),$$

$$\tilde{\Re}_{4v} = \left(\beta^3 \nabla_{3v} - \frac{k_{\mathrm{eff},k}^{(v)}}{EI}\nabla_{4v}\right) \quad (v = 1,\, 2).$$

Appendix B

The coefficient matrices for the "left" end of the beam, $[B_L]$, and those for the "right" end of the beam, $[B_R]$, with the CS, SS and CC boundary conditions were given below.

(1) Clamped–simply supported beam:

$$[B_L] = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ \beta & 0 & \beta & 0 \end{bmatrix} \end{array} \begin{matrix} 1 \\ 2 \end{matrix}, \tag{B.1}$$

$$[B_R] = \begin{array}{c} \begin{matrix} 8n+1 & 8n+2 & 8n+3 & 8n+4 \end{matrix} \\ \begin{bmatrix} \sin(\beta L) & \cos(\beta L) & \sinh(\beta L) & \cosh(\beta L) \\ -\beta^2 \sin(\beta L) & -\beta^2 \cos(\beta L) & \beta^2 \sinh(\beta L) & \beta^2 \cosh(\beta L) \end{bmatrix} \end{array} \begin{matrix} p-1 \\ p \end{matrix}. \tag{B.2}$$

(2) Simply supported–simply supported beam:

$$[B_L] = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & -\beta^2 & 0 & \beta^2 \end{bmatrix} \end{array} \begin{matrix} 1 \\ 2 \end{matrix}, \tag{B.3}$$

$$[B_R] = \begin{array}{c} \begin{matrix} 8n+1 & 8n+2 & 8n+3 & 8n+4 \end{matrix} \\ \begin{bmatrix} \sin(\beta L) & \cos(\beta L) & \sinh(\beta L) & \cosh(\beta L) \\ -\beta^2 \sin(\beta L) & -\beta^2 \cos(\beta L) & \beta^2 \sinh(\beta L) & \beta^2 \cosh(\beta L) \end{bmatrix} \end{array} \begin{matrix} p-1 \\ p \end{matrix}. \tag{B.4}$$

(3) Clamped–clamped beam:

$$[B_L] = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ \beta & 0 & \beta & 0 \end{bmatrix} \end{array} \begin{matrix} 1 \\ 2 \end{matrix}, \tag{B.5}$$

$$[B_R] = \begin{array}{c} \begin{matrix} 8n+1 & 8n+2 & 8n+3 & 8n+4 \end{matrix} \\ \begin{bmatrix} \sin(\beta L) & \cos(\beta L) & \sinh(\beta L) & \cosh(\beta L) \\ \beta \cos(\beta L) & -\beta \sin(\beta L) & \beta \cosh(\beta L) & \beta \sinh(\beta L) \end{bmatrix} \end{array} \begin{matrix} p-1 \\ p \end{matrix}. \tag{B.6}$$

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